SECTION 1: Optimization and Constrained Optimization

Economic Motivation: As you learned in Econ 100A/101A, economists generally assume that individuals seek to make choices that will maximize their utility (optimization) given the limited resources they possess (i.e. the individual's budget or the numbers of hours in a day are constraints).

Unconstrained Optimization: To find the maximum of a function, look to the derivative (the slope of a line or curve) of that function for guidance. If a well-behaved function has an interior optimum, it will be at a point where the derivative is equal to 0.

1. Write down the function you wish to find the optimum of (e.g. the utility function)
2. Take the derivative of the function with respect to the choice variable
3. Set the derivative equal to 0 and rearrange terms to get an expression for the choice variable as a function of everything else

<u>Example:</u> Find the value of x that maximizes $F(x) = 3x - x^2$.

$$F(x) = 3x - x^2$$

$$\frac{dF(x)}{dx} = 3 - 2x$$

$$3 - 2x = 0 \quad \rightarrow \quad 3 = 2x \quad \rightarrow \quad x = \frac{3}{2}$$

However, we rarely work with utility functions of this form. "Normal goods" have the property that "more is better," so the unconstrained maximum for a normal good's utility function would be an infinite amount. Individuals don't have limitless wealth to buy as much of any good as they desire. Therefore, we are often looking at situations where individuals have to divide limited resources between at least 2 goods, which leads us to constrained optimization.

Constrained Optimization: In Econ 131 we will only deal with choices between 2 goods and will generally assume interior solutions. You will not need to worry about Kuhn-Tucker or second order conditions. However, you should check whether 0 is preferred to the interior solution (i.e. is the utility from consuming 0 of one good and spending all wealth on the other higher than the utility from the optimum you found) when we are interested in whether an individual participates or not. I will explicitly remind you when we are working with topics where that is an issue.

Note: Since we are only going to be dealing with choices among 2 goods, you will be able to turn any multivariable problem into a single variable problem with a little bit of algebra.

Mathematical Tools:

General Forms:

$$\max_{x,z} \; U(x,z) \quad s.t.\; w = p_x x + p_z z$$

- Note: When there are only 2 goods to choose between you can re-write the problem as simply choosing how much of one's wealth is spent on one of the goods.
  - Once you know how much is spent on good x, you know that the remainder of the person's budget is spent on z (i.e $w - p_x x$ is spent on z). Thus, in a two good world, the choice boils down to how much to spend on x.
  - Once you have this set up as a single variable equation with the constraint built in you can solve this the same way you solved the unconstrained maximization.

$$w = p_x x + p_z z \quad \rightarrow \quad z = \frac{w - p_x x}{p_z}$$

$$\max_{x,z} \; U(x,z) \quad s.t.\; w = p_x x + p_z z \quad \leftrightarrow \quad \max_{x} \; U\left(x, \frac{w - p_x x}{p_z}\right)$$

1. Now take the full derivative of our re-written utility function with respect to x and set it equal to 0
2. Solve for x
3. Substitute x into $z = \frac{w - p_x x}{p_z}$ to find the optimal z

Example 1: Homer has $20 dollars to spend on beer (b) and pizza slices (s). A beer costs $2 while a pizza slice costs $1. His utility is $U(s,b) = s^{1/2}b^{1/2}$ How many pizza slices and beers should Homer choose to maximize his utility?

1. Write the budget constraint: s + 2b = 20
2. Re-write his budget constraint: s=20-2b
3. Substitute for s in Homer's utility function: $(20-2b)^{1/2}b^{1/2}$
4. Take the derivative with respect to b and set equal to 0.

$$\frac{1}{2}(20-2b)^{-1/2}(-2)b^{1/2} + \frac{1}{2}b^{-1/2}(20-2b)^{1/2} = 0$$

$$\frac{b^{1/2}}{(20-2b)^{1/2}} = \frac{(20-2b)^{1/2}}{2b^{1/2}}$$

5. Solve for b:
   (20-2b) = 2b → 20=4b → b=5

6. Now plug b=5 into s=20-2b to determine how many slices he buys with his remaining wealth.
   s=20-2*5 → s=10

Solution: Homer buys 10 slices of pizza and 5 beers.

Note: Homer chooses to spend $10 on beer and $10 on pizza – i.e. he spends half of his budget on each. Notice that both s and b are raised to the ½ in Homer's utility function. <u>*This is not a coincidence.*</u>

Cobb-Douglas Preferences: Utility functions of the form $U(A,B) = A^{\alpha}B^{\beta}$ with $\alpha + \beta = 1$ are called "Cobb-Douglas" preferences. Functions of this form are used all the time in economics because they are very user friendly and have some nice properties.

- <u>General Form:</u> $U(A,B) = A^{\alpha}B^{\beta}$ with $\alpha + \beta = 1$

- <u>Example:</u> $U(A,B) = A^{0.25}B^{0.75}$

- Utility functions of this form have some very useful properties:
  - They can be re-written as: $U(A,B) = A^{\alpha}B^{1-\alpha}$
  - They represent two normal goods with positive diminishing marginal utility in each good.

$$\frac{\partial U}{\partial A} = \alpha A^{\alpha-1}B^{1-\alpha} \qquad \frac{\partial^2 U}{\partial^2 A} = \alpha(\alpha-1)A^{\alpha-2}B^{1-\alpha}$$

$$\frac{\partial U}{\partial B} = (1-\alpha)A^{\alpha}B^{-\alpha} \qquad \frac{\partial^2 U}{\partial^2 B} = -\alpha(1-\alpha)A^{\alpha}B^{-\alpha-1}$$

  - The optimal allocation of resources across the two goods will have a predictable form.

  - <u>General Form:</u>

Let: p_A=the price of each unit of A
p_B=the price of each unit of B

$$\max_{A,B} U(A,B) = A^{\alpha}B^{1-\alpha} \ s.t. \ p_A A + p_B B = w$$

$$A^* = \frac{\alpha w}{p_A}, \quad B^* = \frac{(1-\alpha)w}{p_B}$$

  - If your constrained optimization is rusty, practice deriving this result.

  - **Economic Implications:**
    1. THIS WILL SAVE YOU TIME ON EXAMS!

    2. This implies that individuals with these types of preferences devote a *constant share of wealth* to each good. The share of wealth devoted to each good corresponds to the exponent on that good in the utility function.
       - In the above example the individual spends αw on A and $(1-\alpha)w$ on B.

    3. Notice, the price of B does NOT appear in the formula for the optimal amount of A. Similarly the price of A appears nowhere in the optimal amount of B.

Let's apply this result to Homer's problem:

- Homer's utility function is: $U(s,b) = s^{1/2}b^{1/2}$
- First: check that Homer's utility function is Cobb-Douglas ½ + ½ =1
- Therefore we know that he spends ½ his budget on pizza and ½ on beer. Therefore he spends $10 on each.
- If pizza slices cost $1, then $10 will buy **10 slices of pizza**
- If beer costs $2, then $10 will buy **5 beers.**
- Thus we get the same answer as before with much less work!

Another useful short-cut: One characteristic of the optimal combination (assuming you are not at a corner) is that the marginal rate of substitution equals the price ratio (which occurs when the indifference curve is tangent to the budget constraint). **Note: This is true in general, not just for Cobb-Douglas preferences.**

$$\frac{MU_A}{MU_B} = \frac{P_A}{P_B} \quad \leftrightarrow \quad \frac{\partial U / \partial A}{\partial U / \partial B} = \frac{P_A}{P_B}$$

Let's return to Homer's problem and solve it using this approach:

$$\frac{\partial U}{\partial s} = \frac{b^{1/2}}{2s^{1/2}} \qquad \frac{\partial U}{\partial b} = \frac{s^{1/2}}{2b^{1/2}} \qquad p_s = 1 \quad p_b = 2$$

$$\frac{\partial U / \partial s}{\partial U / \partial b} = \frac{P_s}{P_b} \quad \rightarrow \quad \left(\frac{b^{1/2}}{2s^{1/2}}\right)\left(\frac{2b^{1/2}}{s^{1/2}}\right) = \frac{1}{2} \quad \rightarrow \quad \frac{2b}{2s} = \frac{1}{2} \quad \rightarrow \quad \frac{b}{s} = \frac{1}{2}$$

Thus, we know that Homer's optimal choice will have 2 slices of pizza for every beer. Now you can plug 2b=s into the budget constraint to find out how many of each Homer will choose.

Note: This was a bit more work than the constant budget share method above, but that method is only available for Cobb-Douglas preferences. As you will see in the following example, the MRS method of finding optimal bundles is particularly helpful when you need to figure out how a change in prices (say due to a tax) will affect demand.

Example 2: Labor Supply

Peggy Hill is a substitute teacher. Every week she has to choose how many hours to work and how many to spend on other activities (leisure). Peggy earns \$10 for every hour she works and therefore has \$10L to spend on all other goods. (There are 168 hours in a week.) The price of all other goods has been normalized to 1. Peggy's utility is:

$U(c,h) = \frac{1}{4}ch - (h-20)^2$ where h is the number of hours of leisure Peggy has.

Solve using $\frac{\partial U / \partial c}{\partial U / \partial h} = \frac{P_c}{P_h}$

First, note that P_h is \$10 (the wage Peggy would have earned had she worked that hour).
Also note that: $c = 10L \quad \rightarrow \quad c = 10(168 - h) = 1680 - 10h$

$$\frac{\partial U}{\partial c} = \frac{1}{4}h \qquad\qquad \frac{\partial U}{\partial h} = \frac{1}{4}c - 2(h-20)$$

$$\frac{0.25h}{0.25c - 2(h-20)} = \frac{1}{10} \quad \rightarrow \quad 2.5h = 0.25c - 2(h-20) \quad \rightarrow \quad 4.5h = 0.25c + 40$$

Now plug in for c using the budget constraint:

$4.5h = 0.25(1680 - 10h) + 40$

$18h = 1680 - 10h + 160 \quad \rightarrow \quad 28h = 1840 \quad \rightarrow \quad h \approx 67$

Peggy chooses 67 hours of leisure. Thus she works 101 hours and earns \$1,010 which she uses to purchase 1,010 units of consumption.

Now, Texas introduces a 20% wage tax. How much does Peggy choose to work now?
Note that the only thing that has changed in $\frac{\partial U / \partial c}{\partial U / \partial h} = \frac{P_c}{P_h}$ is the price ratio, now $\frac{P_c}{P_h} = \frac{1}{8}$
Now, choosing to work for one hour less only involves giving up \$8 of income. The marginal rates of substitution don't change, so they do not need to be re-calculated.

$$\frac{0.25h}{0.25c - 2(h-20)} = \frac{1}{8} \quad \rightarrow \quad 2h = 0.25c - 2h + 40 \quad \rightarrow \quad 16h = c + 160$$

Plug in for c using the budget constraint:

$16h = 8(168 - h) + 160 \quad \rightarrow \quad 24h = 1344 + 160 \quad \rightarrow \quad h \approx 62.67$

Peggy chooses 63 hours of leisure. Thus she works 105 hours and earns \$840 which she uses to purchase 840 units of consumption.